

On Christoffel Type Functions for L_m Extremal Polynomials, II¹

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Asymptotic estimations of the Christoffel type functions for L_m extremal polynomials with an even integer m with respect to the Jacobi weight are established. Also, asymptotic behavior of the zeros of the L_m extremal polynomials and the Cotes numbers of the corresponding Turán quadrature formula is given. © 2001 Academic Press

1. INTRODUCTION AND MAIN RESULTS

Throughout this paper let m be an even integer and $\mathbf{M} := \{0, 2, 4, \dots, m-2\}$. Let μ be a nondecreasing function on \mathbb{R} with infinitely many points of increase such that all moments of $d\mu$ are finite. We call $d\mu$ a measure. For $N \in \mathbb{N}$ let \mathbf{P}_N denote the set of polynomials of degree at most N and \mathbf{P}_N^* the subset of polynomials in \mathbf{P}_N having real zeros only. We agree $\mathbf{P}_0^* = \mathbf{P}_0$. Put $\mathbf{P}_N^*(x) := \{P \in \mathbf{P}_N^* : P(x) = 1\}$ for $x \in \mathbb{R}$. ∂P stands for the exact degree of a polynomial P , i.e., $P \in \mathbf{P}_{\partial P} \setminus \mathbf{P}_{\partial P-1}$. In what follows we denote by c, c_1, \dots positive constants independent of variables and indices, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas. We write $a_n \sim b_n$ if $c_1 \leq a_n/b_n \leq c_2$ holds for every n . The notations $a(x) \sim b(x)$ and $a_n(x) \sim b_n(x)$ have similar meaning. We define the L_m monic extremal polynomials

$$P_n(d\mu, m; x) = x^n + \dots, \quad n = 0, 1, \dots,$$

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for which

$$\int_{\mathbb{R}} P_n(d\mu, m; x)^m d\mu(x) = \min_{P(x)=x^n+\dots} \int_{\mathbb{R}} P(x)^m d\mu(x). \quad (1.1)$$

Then the polynomial $P_n(d\mu, m; x)$ satisfies the L_m orthogonality [10, Theorem 1.11, p. 56]

$$\int_{\mathbb{R}} P_n(d\mu, m; x)^{m-1} q(x) d\mu(x) = 0, \quad q \in \mathbf{P}_{n-1}. \quad (1.2)$$

If μ is absolutely continuous then we will usually write w (called a weight) and $P_n(w, m; x)$ instead of μ' and $P_n(d\mu, m; x)$, respectively. According to Theorem 4 in [1], if $x_{kn} := x_{kn}(d\mu)$ with

$$x_{1n} > x_{2n} > \dots > x_{nn}$$

are the zeros of $P_n(d\mu, m; x)$ then the Gaussian quadrature formula with certain numbers $\lambda_{ik} := \lambda_{iknm}(d\mu)$ (called the Cotes numbers of higher order)

$$\int_{\mathbb{R}} f(x) d\mu(x) = \sum_{i=0}^{m-2} \sum_{k=1}^n \lambda_{ik} f^{(i)}(x_k) \quad (1.3)$$

is exact for all $f \in \mathbf{P}_{nm-1}$.

As we know, the case when $m=2$ is the special case of orthogonal polynomial; it has a long history of study and a classical theory. One of the important contents of this theory is the Christoffel function

$$\lambda_n(d\mu, x) := \min_{P \in \mathbf{P}_{n-1}, P(x)=1} \int_{\mathbb{R}} P(t)^2 d\mu(t), \quad (1.4)$$

which is closely related to the Cotes numbers

$$\lambda_{0kn2}(d\mu) = \lambda_n(d\mu, x_{kn}(d\mu)), \quad k = 1, 2, \dots, n. \quad (1.5)$$

In [9] we introduced and investigated the Christoffel type functions for the L_m extremal polynomials, which are extensions of the Christoffel functions for the classical orthogonal polynomials. Let us give the definition.

Given a fixed $x \in \mathbb{R}$ and $n \in \mathbb{N}$, for $P \in \mathbf{P}_{n-1}$ with $P(x) = 1$ and $j \in \mathbf{M}$ let

$$A_j(P, x; t) := A_{jnm}(P, x; t) := \frac{1}{j!} (t-x)^j B_j(P, x; t) P(t)^m, \\ B_j(P, x; \cdot) \in \mathbf{P}_{m-j-2}, \quad (1.6)$$

satisfy the interpolatory conditions

$$A_j^{(i)}(P, x; x) = \delta_{ij}, \quad i = 0, 1, \dots, m-2. \quad (1.7)$$

Here for simplicity we accept the notation: $f^{(i)}(P, x; t) := \frac{d^i}{dt^i} f(P, x; t)$. It is easy to see that $A_j(P, x; t)$ must exist and be unique. If $P \in \mathbf{P}_{n-1}^*(x)$ then by [9, Lemma 1] we have

$$B_j(P, x; t) = \sum_{i=0}^{m-j-2} b_i(t-x)^i, \quad (1.8)$$

where

$$b_i = b_i(P, x) = \frac{1}{i!} [P(t)^{-m}]_{t=x}^{(i)}, \quad i = 0, 1, \dots,$$

and

$$b_{m-j-2} > 0, \quad B_j(P, x; t) > 0, \quad t \in \mathbb{R}, \quad j \in \mathbf{M}. \quad (1.9)$$

DEFINITION 1. The Christoffel type function $\lambda_{jmm}(d\mu, x)$ ($j \in \mathbf{M}$) with respect to $d\mu$ is defined

$$\lambda_{jmm}(d\mu, x) = \inf_{P \in \mathbf{P}_{n-1}^*(x)} \int_{\mathbb{R}} A_j(P, x; t) d\mu(t).$$

According to (1.9) we conclude $\lambda_{jmm}(d\mu, x) > 0$.

Remark. For $m \geq 4$ the restriction $P \in \mathbf{P}_{n-1}^*(x)$ can not be replaced by the condition $P \in \mathbf{P}_{n-1}$ with $P(x) = 1$, for otherwise the relation (1.9) may be violated and the relation $\lambda_{jmm}(d\mu, x) = -\infty$ may occur. For example choose $P(t) = 1 + d(t-x)^2$ with a number $d > 0$. By a simple calculation we see

$$A_{m-4}(P, x; t) = \frac{1}{(m-4)!} (t-x)^{m-4} [1 - md(t-x)^2] P(t)^m, \quad m \geq 4.$$

Thus by the mean value theorem for integration with a certain number τ

$$\begin{aligned} & \int_{\mathbb{R}} A_{m-4}(P, x; t) d\mu(t) \\ &= \frac{1}{(m-4)!} P(\tau)^m \int_{\mathbb{R}} (t-x)^{m-4} [1 - md(t-x)^2] d\mu(t), \end{aligned}$$

which tends to $-\infty$ as $d \rightarrow +\infty$, because $P(\tau) \geq 1$.

In this paper we will give asymptotic estimations of $\lambda_{jnm}(w, x)$ for a weight $w \sim w^{(\alpha, \beta)}$, where

$$\begin{aligned} w^{(\alpha, \beta)}(x) &= (1-x)^\alpha (1+x)^\beta, & |x| < 1, \\ w^{(\alpha, \beta)}(x) &= 0, & |x| \geq 1, \quad \alpha, \beta > -1. \end{aligned}$$

Based on this result we will determine asymptotic behavior of the zeros $x_{kn}(w)$ and the Cotes numbers $\lambda_{jknm}(w)$ for $w \sim w^{(\alpha, \beta)}$. The main results are as follows.

THEOREM 1. *If $j \in \mathbf{M}$ and*

$$w \sim w^{(\alpha, \beta)}, \tag{1.10}$$

then, with the constants associated with the symbol \sim depending on m and w ,

$$\begin{aligned} \lambda_{jnm}(w, x) &\sim \lambda_n(w, x) \Delta_n(x)^j \\ &\sim \frac{1}{n} w_n^{(\alpha, \beta)}(x) \Delta_n(x)^j, \quad x \in [-1, 1], \end{aligned} \tag{1.11}$$

where

$$\Delta_n(x) := \frac{(1-x^2)^{1/2}}{n} + \frac{1}{n^2}$$

and

$$w_n^{(\gamma, \delta)}(x) := \left[(1-x)^{1/2} + \frac{1}{n} \right]^{2\gamma+1} \left[(1+x)^{1/2} + \frac{1}{n} \right]^{2\delta+1}, \quad \gamma, \delta \in \mathbb{R}.$$

Clearly,

$$\begin{aligned} w_n^{(\alpha, \alpha)}(x) &= \left\{ (1-x^2)^{1/2} + \frac{1}{n} [(1-x)^{1/2} + (1+x)^{1/2}] + \frac{1}{n^2} \right\}^{2\alpha+1} \\ &\sim \left[(1-x^2)^{1/2} + \frac{1}{n} \right]^{2\alpha+1} = [n\Delta_n(x)]^{2\alpha+1}. \end{aligned}$$

Thus, as a direct consequence of Theorem 1, we have

COROLLARY 1. *If $j \in \mathbf{M}$ and $w \sim w^{(\alpha, \alpha)}$, then, with the constants associated with the symbol \sim depending on m and w ,*

$$\lambda_{jnm}(w, x) \sim n^{2\alpha} \Delta_n(x)^{2\alpha + j + 1}, \quad x \in [-1, 1]. \quad (1.12)$$

By the definition of λ_{jnm} from (1.12) we obtain

COROLLARY 2. *If $j \in \mathbf{M}$ and*

$$w(x) \leq (\text{or } \geq) \frac{c}{\sqrt{1-x^2}}, \quad x \in (-1, 1), \quad (1.13)$$

then

$$\lambda_{jnm}(w, x) \leq (\text{or } \geq) c_1 n^{-1} \Delta_n(x)^j, \quad x \in [-1, 1]; \quad (1.14)$$

if $j \in \mathbf{M}$ and

$$w(x) \leq (\text{or } \geq) c, \quad x \in [-1, 1], \quad (1.15)$$

then

$$\lambda_{jnm}(w, x) \leq (\text{or } \geq) c_2 \Delta_n(x)^{j+1}, \quad x \in [-1, 1]. \quad (1.16)$$

THEOREM 2. *Let*

$$x_{kn}(w) = \cos \theta_{kn}(w), \quad k = 0, 1, \dots, n+1, \quad x_{0n}(w) = 1, \quad x_{n+1,n}(w) = -1.$$

If (1.10) is true, then, with the constants associated with the symbol \sim depending on m and w ,

$$\theta_{k+1,n}(w) - \theta_{kn}(w) \sim \frac{1}{n}, \quad k = 0, 1, \dots, n. \quad (1.17)$$

As an immediate consequence of Theorem 2 we state

COROLLARY 3. *If (1.10) is true, then, with the constants associated with the symbol \sim depending on m and w ,*

$$\theta \sim \theta_{kn}(w), \quad \theta \in [\theta_{kn}(w), \theta_{k+1,n}(w)], \quad k = 1, 2, \dots, n \quad (1.18)$$

and

$$x_{kn}(w) - x_{k+1,n}(w) \sim \Delta_n(x_{kn}(w)), \quad k = 0, 1, \dots, n. \quad (1.19)$$

The special cases when $m=2$ of Theorems 1 and 2 can be found in [6, Lemma 1] (Lemma B below) and [6, Theorem 3], respectively.

THEOREM 3. *Let $j \in \mathbf{M}$ be fixed. Then (1.10) is valid if and only if both (1.11) and (1.17) are valid, in which the constants associated with the symbol \sim depend on m and w .*

For the Chebyshev weight $v := w^{(-1/2, -1/2)}$ P. Turán raised the following problem [12, p. 47]:

Problem 26. Give an explicit formula for $\lambda_{jknm}(v)$ and determine its asymptotic behavior as $n \rightarrow \infty$.

The author [7] gives an answer to this problem. In this paper the following theorem is to give asymptotic behavior of $\lambda_{jknm}(w)$ as $n \rightarrow \infty$ for $w \sim w^{(\alpha, \beta)}$.

THEOREM 4. *If (1.10) is true, then, with the constants associated with the symbol \sim depending on m and w ,*

$$\begin{aligned} \lambda_{jknm}(w) &\sim \lambda_n(w, x_{kn}(w)) \Delta_n(x_{kn}(w))^j \\ &\sim \frac{1}{n} w_n^{(\alpha, \beta)}(x_{kn}(w)) \Delta_n(x_{kn}(w))^j, \quad j \in \mathbf{M}, \\ |\lambda_{jknm}(w)| &\leq c \lambda_n(w, x_{kn}(w)) \Delta_n(x_{kn}(w))^j \\ &\sim \frac{1}{n} w_n^{(\alpha, \beta)}(x_{kn}(w)) \Delta_n(x_{kn}(w))^j, \quad j \notin \mathbf{M}. \end{aligned} \tag{1.20}$$

These interesting results extend the results for the Jacobi polynomials ($m = 2$) and will play important role in various applications of L_m extremal polynomials, which will be considered in forthcoming papers. But a question whether or not the results in this paper can be extended to weight functions other than Jacobi weights is still open.

In the next section some auxiliary lemmas are established and in the last section the proofs of the theorems are given.

2. AUXILIARY LEMMAS

First we state some known results needed later. As usual, $T_n(x)$ stands for the n th Chebyshev polynomial of the first kind.

LEMMA A [9, Theorems 1 and 2, Corollary 1]. *There exists a unique polynomial $P \in \mathbf{P}_{n-1}^*(x)$ such that*

$$\int_{\mathbb{R}} A_j(P, x; t) d\mu(t) = \lambda_{jnm}(d\mu, x) \tag{2.1}$$

holds for every $j \in \mathbf{M}$; in particular, if $x = x_{kn}(d\mu)$ then

$$P(t) = \ell_k(t) := \prod_{i \neq k} \frac{t - x_{in}(d\mu)}{x_{kn}(d\mu) - x_{in}(d\mu)}. \quad (2.2)$$

Moreover,

$$\lambda_{jknm}(d\mu) = \lambda_{jnm}(d\mu, x_{kn}(d\mu)), \quad k = 1, 2, \dots, n, \quad j \in \mathbf{M}, \quad (2.3)$$

and

$$\lambda_{0n2}(d\mu, x) = \lambda_n(d\mu, x).$$

LEMMA B [6, Lemma 1]. *If (1.10) is true, then*

$$\lambda_n(w, x) \sim \frac{1}{n} w_n^{(\alpha, \beta)}(x), \quad |x| \leq 1. \quad (2.4)$$

LEMMA C [5, Lemma 6.3.8, p. 108]. *Let*

$$K_n(v, x; t) = \frac{T_n(x) T_{n-1}(t) - T_{n-1}(x) T_n(t)}{\pi(x-t)}, \quad n \geq 2.$$

Then

$$|K_n(v, x; t)| \leq c \min \left\{ n, \frac{(1-x^2)^{1/2} + (1-t^2)^{1/2}}{|x-t|} \right\},$$

$$x, t \in [-1, 1]. \quad (2.5)$$

LEMMA D [5, Lemma 6.3.10, p. 109]. *Let $w = w^{(\alpha, \beta)}$ and*

$$\pi_{n-1}(x; t) := \frac{K_n(v, x; t)}{K_n(v, x; x)}.$$

Then there exists a positive integer $N_1 = N_1(\alpha, \beta)$ such that for every fixed integer $N \geq N_1$

$$\int_{-1}^1 |\pi_{n-1}(x; t)|^N w(t) dt \leq \frac{c}{n} w_n^{(\alpha, \beta)}(x), \quad x \in [-1, 1], \quad n \in \mathbb{N}. \quad (2.6)$$

LEMMA E [2, (3.7), p. 102; p. 104]. *We have*

$$|\pi_{n-1}(x; t)| \leq 4, \quad n \geq 3, \quad x, t \in [-1, 1], \quad (2.7)$$

and

$$K_n(v, x; x) \sim n, \quad |x| \leq 1. \quad (2.8)$$

LEMMA F [3, Theorem 1]. *We have the generalized Markov–Stieltjes inequality*

$$\sum_{k=r+1}^n \lambda_{0knm}(d\mu) \leq \int_{-\infty}^{x_m(d\mu)} d\mu(t) \leq \sum_{k=r}^n \lambda_{0knm}(d\mu). \quad (2.9)$$

LEMMA G [4]. *Let $P \in \mathbf{P}_n$. Then*

$$\begin{aligned} \max_{|x| \leq 1} |P'(x) w^{(\gamma+(1/2), \delta+(1/2))}(x)| \\ \leq cn \max_{|x| \leq 1} |P(x) w_n^{(\gamma, \delta)}(x)|, \quad \gamma, \delta \in \mathbb{R}, \end{aligned} \quad (2.10)$$

$$\begin{aligned} \max_{|x| \leq 1 - (c_1/n^2)} |P(x) w^{(\alpha+(1/2), \beta+(1/2))}(x)| \\ \leq cn \int_{-1}^1 |P(x)| w^{(\alpha, \beta)}(x) dx, \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \int_{-1}^1 |P'(x)| w^{(\alpha+(1/2), \beta+(1/2))}(x) dx \\ \leq cn \int_{-1}^1 |P(x)| w^{(\alpha, \beta)}(x) dx. \end{aligned} \quad (2.12)$$

A. Markov proved an important result concerning the dependence of the zeros of $P_n(w, 2; x)$ on a parameter τ which appears in the weight function $w(x) = w(\tau, x)$ (see, say, [11, Theorem 6.12.1, p. 111]). Using his idea we can deduce an analogue for L_m extremal polynomials. But for our applications later we relax the original assumption about continuity of $w(\tau, x)$ and $w_\tau(\tau, x)$ with respect to x and τ and assume continuity of $w(\tau, x)$ and $w_\tau(\tau, x)$ with respect to τ only. That is the following

LEMMA 1. *Let $w(\tau, x)$ be a weight on $[a, b]$ depending on a parameter τ such that $w(\tau, x)$ is positive and continuous with respect to τ for $x \in (a, b)$, $\tau \in (\tau_1, \tau_2)$. Also, assume the existence and continuity (with respect to τ) of*

the partial derivative $w_\tau(\tau, x)$ for $x \in (a, b)$, $\tau \in (\tau_1, \tau_2)$, and the convergence of the integrals

$$\int_b^a x^\nu w_\tau(\tau, x) dx, \quad \nu = 0, 1, \dots, 2n-1,$$

uniformly in every closed interval $[\tau', \tau''] \subset (\tau_1, \tau_2)$. If w_τ/w is an increasing function of x in (a, b) , then the ν th zero $x_\nu(\tau) = x_\nu(w(\tau))$ (for a fixed index ν) is an increasing function of τ .

Proof. Differentiating the formula (1.3) with $d\mu(x) = w(\tau, x) dx$ with respect to τ , we obtain

$$\begin{aligned} & \int_b^a f(x) w_\tau(\tau, x) dx \\ &= \sum_{i=0}^{m-2} \sum_{k=1}^n [\lambda_{ik}(\tau) f^{(i+1)}(x_k(\tau)) x'_k(\tau) + \lambda'_{ik}(\tau) f^{(i)}(x_k(\tau))], \end{aligned}$$

where

$$x'_k(\tau) = \frac{\partial x_k(w(\tau))}{\partial \tau}, \quad \lambda'_{ik}(\tau) = \frac{\partial \lambda_{ik}(w(\tau))}{\partial \tau}.$$

Substituting

$$f(x) = \frac{P_n(w(\tau), m; x)^m}{(m-1)! (x - x_k(\tau))}$$

into the above formula and observing $f^{(m-1)}(x_k(\tau)) = P'_n(w(\tau), m; x_k(\tau))^m$, here

$$P'_n(w(\tau), m; x_k(\tau)) = \frac{d}{dx} P_n(w(\tau), m; x)|_{x=x_k(\tau)},$$

we get

$$\int_b^a \frac{P_n(w(\tau), m; x)^m}{(m-1)! (x - x_k(\tau))} w_\tau(\tau, x) dx = \lambda_{m-2, k}(\tau) P'_n(w(\tau), m; x_k(\tau))^m x'_k(\tau).$$

Using L_m orthogonality (1.2) we see

$$\begin{aligned} & \int_b^a \frac{P_n(w(\tau), m; x)^m}{x - x_k(\tau)} w_\tau(\tau, x) dx \\ &= \int_b^a \frac{P_n(w(\tau), m; x)^m}{x - x_k(\tau)} \left[w_\tau(\tau, x) - \frac{w_\tau(\tau, x_k(\tau))}{w(\tau, x_k(\tau))} w(\tau, x) \right] dx \\ &= \int_b^a \frac{P_n(w(\tau), m; x)^m}{x - x_k(\tau)} \left[\frac{w_\tau(\tau, x)}{w(\tau, x)} - \frac{w_\tau(\tau, x_k(\tau))}{w(\tau, x_k(\tau))} \right] w(\tau, x) dx > 0, \end{aligned}$$

because the difference in the square brackets has the same sign as $x - x_k(\tau)$ by the assumption. This proves $x'_k(\tau) > 0$, because $\lambda_{m-2, k}(\tau) > 0$ by (1.6), (1.9), (2.1), and (2.3). ■

As consequences of this lemma we state two lemmas which are extensions of [11, Theorems 6.12.2 and 6.21.1, pp. 112–117] and may be derived by the same arguments as that of those theorems in [11].

LEMMA 2. *Let $w(x)$ and $W(x)$ be weights on $[a, b]$, both positive in (a, b) . Let $W(x)/w(x)$ be increasing. If $\{x_k\}$ and $\{X_k\}$ denote the zeros of the corresponding L_m extremal polynomials of degree n in decreasing order, then*

$$x_k < X_k, \quad k = 1, 2, \dots, n.$$

LEMMA 3. *Let $w(x)$ be a weight on $[-1, 1]$ and positive in $(-1, 1)$. Let*

$$w_1(\tau, x) = w(x)(1-x)^\tau \quad (\text{or } w_2(\tau, x) = w(x)(1+x)^\tau) \quad (2.13)$$

be a weight on $[-1, 1]$ with a parameter τ for $\tau \in (\tau_1, \tau_2)$ and $x_k(w_1(\tau))$ (or $x_k(w_2(\tau))$) be the zeros of $P_n(w_1(\tau), m; x)$ (or $P_n(w_2(\tau), m; x)$). Assume that the convergence of the integrals

$$\begin{aligned} & \int_{-1}^1 x^v \ln(1-x) w_1(\tau, x) dx \\ & \left(\text{or } \int_{-1}^1 x^v \ln(1+x) w_2(\tau, x) dx \right), \quad v = 0, 1, \dots, 2n-1, \end{aligned}$$

hold uniformly in every closed interval $[\tau', \tau''] \subset (\tau_1, \tau_2)$. Then

$$\frac{\partial x_k(w_1)}{\partial \tau} < 0 \quad \left(\text{or } \frac{\partial x_k(w_2)}{\partial \tau} > 0 \right), \quad k = 1, 2, \dots, n. \quad (2.14)$$

LEMMA 4. *With the notation of (2.13) if (1.10) is true, then*

$$\begin{aligned} P_n(w_1(-1-\alpha), m; x) &:= \lim_{\tau \rightarrow -1-\alpha+0} P_n(w_1(\tau), m; x) \\ &= (x-1) P_{n-1}(w_1(m-1-\alpha), m; x). \end{aligned} \quad (2.15)$$

Proof. According to Lemma 3 the limit

$$\lim_{\tau \rightarrow -1-\alpha+0} P_n(w_1(\tau), m; x)$$

exists and the convergence is uniform with respect to x in $[-1, 1]$. By the definition of $P_n(w_1(\tau), m; x)$ for $-1-\alpha < \tau \leq 0$

$$\begin{aligned} \int_{-1}^1 P_n(w_1(\tau), m; x)^m w_1(\tau, x) dx &\leq \int_{-1}^1 [(x-1)^n]^m w_1(\tau, x) dx \\ &\leq \int_{-1}^1 (1-x)^{mm} w(x) \left(\frac{1-x}{2}\right)^\tau dx \\ &\leq 2^{\alpha+1} \int_{-1}^1 (1-x)^{mm-1-\alpha} w(x) dx \\ &= c < \infty. \end{aligned}$$

Thus

$$\lim_{\tau \rightarrow -1-\alpha+0} x_1(w_1(\tau)) = 1. \quad (2.16)$$

Meanwhile by (1.2)

$$\int_{-1}^1 P_n(w_1(\tau), m; x)^{m-1} q(x) w_1(\tau, x) dx = 0, \quad \forall q \in \mathbf{P}_{n-1}. \quad (2.17)$$

For an arbitrary $Q \in \mathbf{P}_{n-2}$ substituting $q(x) = (x-1) Q(x)$ into (2.17), we obtain

$$\int_{-1}^1 P_n(w_1(\tau), m; x)^{m-1} Q(x) w_1(\tau+1, x) dx = 0.$$

As $\tau \rightarrow -1-\alpha+0$ the above equation gives

$$\int_{-1}^1 P_n(w_1(-1-\alpha), m; x)^{m-1} Q(x) w_1(-\alpha, x) dx = 0,$$

or by (2.16) equivalently,

$$\int_{-1}^1 \left[\frac{P_n(w_1(-1-\alpha), m; x)}{x-1} \right]^{m-1} \times Q(x) w_1(m-1-\alpha, x) dx = 0, \quad \forall Q \in \mathbf{P}_{n-2},$$

which by (1.2) means

$$\begin{aligned} & \int_{-1}^1 \left[\frac{P_n(w_1(-1-\alpha), m; x)}{x-1} \right]^m w_1(m-1-\alpha, x) dx \\ &= \min_{P(x)=x^{n-1}+\dots} \int_{-1}^1 P(x)^m w_1(m-1-\alpha, x) dx \\ &= \int_{-1}^1 P_{n-1}(w_1(m-1-\alpha), m; x)^m w_1(m-1-\alpha, x) dx. \end{aligned}$$

Thus (2.15) follows. ■

LEMMA 5. *If $d \geq 0$ and $y > x \geq 0$ then*

$$\frac{(y-x)(y^d+x^d)}{y^{d+1}-x^{d+1}} \leq 2.$$

Proof. In fact, we have

$$\begin{aligned} 2(y^{d+1}-x^{d+1})-(y-x)(y^d+x^d) &= y^{d+1}-x^{d+1}+xy^d-x^d y \\ &= (y+x)(y^d-x^d) \geq 0, \end{aligned}$$

which is equivalent to our assertion. ■

By [2, Theorem 3.1, p. 19] the polynomial $\pi_{n-1}(x; t)$ in t given in Lemma D has $n^* \geq n-2$ real and simple zeros; hence

$$\pi_{n-1}(x; \cdot) \in \mathbf{P}_{n-1}^*(x). \quad (2.18)$$

LEMMA 6. *We have*

$$|b_i(\pi_{n-1}(x; \cdot); x)| \leq cA_n(x)^{-i}, \quad |x| \leq 1, \quad i=0, 1, \dots \quad (2.19)$$

Proof. The estimation (2.7) implies

$$\left| \frac{d^i}{dt^i} [\pi_{n-1}(x; t)^m]_{t=x} \right| \leq c A_n(x)^{-i}, \quad |x| \leq 1, \quad i = 0, 1, \dots$$

Then by the recurrence relation

$$\begin{aligned} & b_i(\pi_{n-1}(x; \cdot); x) \\ &= - \sum_{v=0}^{i-1} b_v(\pi_{n-1}(x; \cdot), x) \\ & \quad \times \left\{ \frac{1}{(i-v)!} \frac{d^{i-v}}{dt^{i-v}} [\pi_{n-1}(x; t)^m]_{t=x} \right\}, \quad i = 1, 2, \dots \end{aligned} \quad (2.20)$$

we can get (2.19), where (2.20) may be derived by Newton–Leibniz rule from the identity

$$[\pi_{n-1}(x; t)^{-m} \pi_{n-1}(x; t)^m]_{t=x}^{(i)} = 0, \quad i \geq 1. \quad \blacksquare$$

LEMMA 7. Let $d_k = \max\{|x_{kn} - x_{k-1, n}|, |x_{kn} - x_{k+1, n}|\}$, $1 \leq k \leq n$, $x_{0n} = -x_{n+1, n} = 1$. Then for $j \in \mathbf{M}$ and $i > j$

$$|\lambda_{iknm}(d\mu)| \leq d_k^{i-j} \lambda_{jknm}(d\mu), \quad 1 \leq k \leq n, \quad n \in \mathbb{N}. \quad (2.21)$$

Proof. Substituting $A_i(\ell_k, x_k; t)$, defined by (1.6) and (2.2), into (1.3) yields

$$\lambda_{ik}(d\mu) = \int_{\mathbb{R}} A_i(\ell_k, x_k; t) d\mu(t), \quad 0 \leq i \leq m-2, \quad 1 \leq k \leq n.$$

Thus using an estimation given by the author [8, (2.22)]

$$|A_i(\ell_k, x_k; t)| \leq d_k^{i-j} A_j(\ell_k, x_k; t), \quad t \in \mathbb{R},$$

for $i > j$ and $j \in \mathbf{M}$, we directly get (2.21). \blacksquare

LEMMA 8. Let $P_n(x) = P_n(d\mu, p; x)$ satisfy (1.1) with m replaced by $p \geq 1$ and let x_{kn} be its zeros. Then

$$x_{1n} > x_{1, n-1} > x_{2n} > x_{2, n-1} > \dots > x_{n-1, n} > x_{n-1, n-1} > x_{nm}. \quad (2.22)$$

Proof. By the characterization theorem of L_p approximation [10, Theorem 1.11, p. 56]

$$\int_{\mathbb{R}} |P_n(x)|^{p-2} P_n(x) q(x) d\mu(x) = 0, \quad \forall q \in \mathbf{P}_{n-1}$$

and

$$\int_{\mathbb{R}} |P_{n-1}(x)|^{p-2} P_{n-1}(x) q(x) d\mu(x) = 0, \quad \forall q \in \mathbf{P}_{n-2}.$$

Thus

$$\begin{aligned} \int_{\mathbb{R}} [t |tP_n(x)|^{p-2} P_n(x) - |P_{n-1}(x)|^{p-2} P_{n-1}(x)] q(x) d\mu(x) &= 0, \\ \forall q \in \mathbf{P}_{n-2}, \quad t \in \mathbb{R}. \end{aligned} \quad (2.23)$$

With the notations

$$f_t(x) = tP_n(x) - P_{n-1}(x) \quad (2.24)$$

and

$$W(x) = \frac{t |tP_n(x)|^{p-2} P_n(x) - |P_{n-1}(x)|^{p-2} P_{n-1}(x)}{f_t(x)}$$

(2.23) becomes

$$\int_{\mathbb{R}} f_t(x) q(x) W(x) d\mu(x) = 0, \quad \forall q \in \mathbf{P}_{n-2}, \quad t \in \mathbb{R}. \quad (2.25)$$

It is easy to check that $W(x) \geq 0$ in \mathbb{R} . In fact, if we write W as

$$W(x) = \frac{\{\operatorname{sgn}[tP_n(x)]\} |tP_n(x)|^{p-1} - \{\operatorname{sgn} P_{n-1}(x)\} |P_{n-1}(x)|^{p-1}}{\{\operatorname{sgn}[tP_n(x)]\} |tP_n(x)| - \{\operatorname{sgn} P_{n-1}(x)\} |P_{n-1}(x)|}$$

and observe that $|tP_n(x)| \geq (\leq) |P_{n-1}(x)|$ implies $|tP_n(x)|^{p-1} \geq (\leq) |P_{n-1}(x)|^{p-1}$ ($p \geq 1$) then we can assert our conclusion. In order to prove our lemma it suffices to show that each of the open intervals $(x_{kn}, x_{k+1,n})$, $k=1, 2, \dots, n-1$, contains one point of $x_{k,n-1}$'s. Now suppose to the contrary that there would be an interval $\Delta = (x_{rn}, x_{r+1,n})$, $1 \leq r \leq n-1$, which contains no any point of $x_{k,n-1}$'s. Then both the polynomials $P_n(x)$ and $P_{n-1}(x)$ are of constant sign in Δ . Thus we can choose the number t such that $f_t(x)$ has a zero of even order. But for this number t the equation (2.25) implies that $f_t(x)$ has at least $n-1$ sign changes and hence has at

least $n - 1$ zeros of odd order. So the number of the zeros of $f_t(x)$ (counting multiplicities) is more than $n + 1$, a contradiction. ■

3. PROOFS OF THEOREMS

3.1. Proof of Theorem 1

By the definition (1.4)

$$\lambda_{mn/2}(w, x) = \min_{Q \in \mathbf{P}_{(mn/2)-1}} \frac{1}{Q(x)^2} \int_{-1}^1 Q(t)^2 w(t) dt. \quad (3.1)$$

Then

$$Q(x)^2 \leq \lambda_{mn/2}(w, x)^{-1} \int_{-1}^1 Q(t)^2 w(t) dt, \quad Q \in \mathbf{P}_{(mn/2)-1}. \quad (3.2)$$

Let $P \in \mathbf{P}_{n-1}^*(x)$ satisfy (2.1) with $d\mu(t) = w(t) dt$. Since $A_j(P, x; t) \geq 0$ in \mathbb{R} by (1.9), according to [11, Theorem 1.21.2, p. 5] it may be written as

$$A_j(P, x; t) = R(t)^2 + Q(t)^2, \quad R, Q \in \mathbf{P}_{(mn/2)-1}.$$

Thus by (3.2), (2.1), and (2.4)

$$\begin{aligned} A_j(P, x; t) &= R(t)^2 + Q(t)^2 \leq \lambda_{mn/2}(w, t)^{-1} \int_{-1}^1 [R(s)^2 + Q(s)^2] w(s) ds \\ &= \lambda_{mn/2}(w, t)^{-1} \int_{-1}^1 A_j(P, x; s) w(s) ds \\ &= \lambda_{mn/2}(w, t)^{-1} \lambda_{jnm}(w, x) \\ &\leq cn w_n^{(\alpha, \beta)}(t)^{-1} \lambda_{jnm}(w, x), \quad |t| \leq 1. \end{aligned} \quad (3.3)$$

Applying (2.10) j times and using the above inequality, we obtain

$$\begin{aligned} \max_{|t| \leq 1} |A_j^{(j)}(P, x; t) w_n^{(\alpha + (j/2), \beta + (j/2))}(t)| \\ \leq cn^j \max_{|t| \leq 1} |A_j(P, x; t) w_n^{(\alpha, \beta)}(t)| \leq cn^{j+1} \lambda_{jnm}(w, x). \end{aligned}$$

By (1.7) the above relation gives

$$w_n^{(\alpha + (j/2), \beta + (j/2))}(x) \leq cn^{j+1} \lambda_{jnm}(w, x).$$

Hence by (2.4)

$$\lambda_{jnm}(w, x) \geq cn^{-j-1} w_n^{(\alpha+(j/2), \beta+(j/2))}(x) \geq c\lambda_n(w, x) \Delta_n(x)^j.$$

In order to prove the second half of the theorem for each i , $0 \leq i \leq m-2$, by Lemma D choose N_i so large that

$$\begin{aligned} & \int_{-1}^1 |\pi_{n-1}(x; t)|^N w^{(\alpha+(i/2), \beta+(i/2))}(t) dt \\ & \leq \frac{c}{n} w_n^{(\alpha+(i/2), \beta+(i/2))}(x), \quad x \in [-1, 1], \quad n \in \mathbb{N}, \end{aligned} \quad (3.4)$$

holds for every fixed integer $N \geq N_i = N_i(\alpha, \beta)$. Put

$$p = \left\lceil \frac{1 + \max_{0 \leq i \leq m-2} N_i}{2} \right\rceil, \quad r = 1 + \left\lceil \frac{n-1}{p} \right\rceil,$$

and

$$P(t) = \pi_{r-1}(x; t)^p. \quad (3.5)$$

Then $\partial P \leq p(r-1) \leq n-1$, $2p \geq N_i$, $i = 0, 1, \dots, m-2$, and hence by (3.4)

$$\begin{aligned} & \int_{-1}^1 P(t)^2 w^{(\alpha+(i/2), \beta+(i/2))}(t) dt \\ & = \int_{-1}^1 |\pi_{r-1}(x; t)|^{2p} w^{(\alpha+(i/2), \beta+(i/2))}(t) dt \\ & \leq \frac{c}{r} w_r^{(\alpha+(i/2), \beta+(i/2))}(x) \\ & \leq \frac{c}{n} w_n^{(\alpha+(i/2), \beta+(i/2))}(x), \quad i = 0, 1, \dots, m-2, \end{aligned} \quad (3.6)$$

where $c = c(\alpha, \beta, m)$.

By the definition of $\lambda_{jnm}(w, x)$, recalling (2.18) and $\partial P \leq n-1$, it follows from (1.8) and (3.5) that

$$\begin{aligned} \lambda_{jnm}(w, x) & \leq c \int_{-1}^1 (t-x)^j B_j(P, x; t) P(t)^m w^{(\alpha, \beta)}(t) dt \\ & \leq c \sum_{i=0}^{m-2-j} |b_i(P; x)| \int_{-1}^1 |t-x|^{i+j} \pi_{r-1}(x; t)^{pm} w^{(\alpha, \beta)}(t) dt. \end{aligned} \quad (3.7)$$

Let us estimate

$$S_i = \int_{-1}^1 |t-x|^{i+j} \pi_{r-1}(x; t)^{pm} w^{(\alpha, \beta)}(t) dt, \quad 0 \leq i \leq m-2-j.$$

Using (2.5), (2.8), and (3.6) we get

$$\begin{aligned} S_i &= \int_{-1}^1 \pi_{r-1}(x; t)^{2p} [|t-x| K_r(v, x; t)]^{i+j} \\ &\quad \times \frac{K_r(v, x; t)^{p(m-2)-i-j}}{K_r(v, x; x)^{p(m-2)}} w^{(\alpha, \beta)}(t) dt \\ &\leq c \int_{-1}^1 \pi_{r-1}(x; t)^{2p} [(1-x^2)^{1/2} + (1-t^2)^{1/2}]^{i+j} \\ &\quad \times \frac{r^{p(m-2)-i-j}}{r^{p(m-2)}} w^{(\alpha, \beta)}(t) dt \\ &= cr^{-i-j} \sum_{v=0}^{i+j} \binom{i+j}{v} (1-x^2)^{(i+j-v)/2} \\ &\quad \times \int_{-1}^1 \pi_{r-1}(x; t)^{2p} (1-t^2)^{v/2} w^{(\alpha, \beta)}(t) dt \\ &\leq cn^{-i-j} \sum_{v=0}^{i+j} \binom{i+j}{v} (1-x^2)^{(i+j-v)/2} n^{-1} w_n^{(\alpha+(v/2), \beta+(v/2))}(x) \\ &= cn^{-i-j-1} w_n^{(\alpha, \beta)}(x) \sum_{v=0}^{i+j} \binom{i+j}{v} [(1-x^2)^{1/2}]^{i+j-v} \left[(1-x^2)^{1/2} + \frac{1}{n} \right]^v \\ &= cn^{-i-j-1} w_n^{(\alpha, \beta)}(x) \left[2(1-x^2)^{1/2} + \frac{1}{n} \right]^{i+j} \\ &\leq cn^{-1} w_n^{(\alpha, \beta)}(x) \Delta_n(x)^{i+j}. \end{aligned}$$

By (2.4)

$$S_i \leq c \lambda_n(w^{(\alpha, \beta)}, x) \Delta_n(x)^{i+j} \leq c \lambda_n(w, x) \Delta_n(x)^{i+j},$$

which, coupled with (2.19) and (3.7), yields

$$\lambda_{jmm}(w, x) \leq c \lambda_n(w, x) \Delta_n(x)^j, \quad x \in [-1, 1], \quad j \in \mathbf{M}. \quad \blacksquare$$

3.2 Proof of Theorem 2

The proof follows and properly modifies ideas of P. Nevai in [6]. We break the proof into three claims.

Claim 1. $\theta_1 \geq \frac{c}{n}$.

By (1.3), (2.2), and (2.3) with $d\mu(t) = w(t) dt$

$$\int_{-1}^1 (1-t) A_{m-2}(\ell_1, x_1; t) w(t) dt = (1-x_1) \lambda_{m-2, n, m}(w, x_1).$$

On the other hand, by the definition of $\lambda_{m-2, n, m}(w_1(1), x)$ with the notation (2.13) we have

$$\begin{aligned} \int_{-1}^1 (1-t) A_{m-2}(\ell_1, x_1; t) w(t) dt &= \int_{-1}^1 A_{m-2}(\ell_1, x_1; t) w_1(1, t) dt \\ &\geq \lambda_{m-2, n, m}(w_1(1), x_1). \end{aligned}$$

Hence

$$(1-x_1) \lambda_{m-2, n, m}(w, x_1) \geq \lambda_{m-2, n, m}(w_1(1), x_1).$$

By (1.11) we get

$$(1-x_1)^{1/2} \geq \frac{c}{n},$$

or equivalently,

$$\theta_1 \geq \frac{c}{n}.$$

Claim 2. $\theta_{k+1} - \theta_k \leq \frac{c}{n}$, $k = 0, 1, \dots, n$.

We shall deal with the case when $x_{k+1} \geq -1/2$, the case when $x_{k+1} < -1/2$ can be treated similarly.

By (2.9), (2.3), and (1.11) ($\lambda_{00} := \lambda_{0, n+1} := 0$)

$$\begin{aligned} \int_{x_{k+1}}^{x_k} w(t) dt &\leq \lambda_{0, k+1} + \lambda_{0k} \\ &\leq \frac{c}{n} \left\{ \left[(1-x_{k+1})^{1/2} + \frac{1}{n} \right]^{2\alpha+1} + \left[(1-x_k)^{1/2} + \frac{1}{n} \right]^{2\alpha+1} \right\}. \end{aligned}$$

Meanwhile

$$\begin{aligned} \int_{x_{k+1}}^{x_k} w(t) dt &\geq c \int_{x_{k+1}}^{x_k} (1-t)^\alpha dt \\ &= \frac{c}{\alpha+1} [(1-x_{k+1})^{\alpha+1} - (1-x_k)^{\alpha+1}]. \end{aligned}$$

Thus

$$\begin{aligned} & (1-x_{k+1})^{\alpha+1} - (1-x_k)^{\alpha+1} \\ & \leq \frac{c}{n} \left\{ \left[(1-x_{k+1})^{1/2} + \frac{1}{n} \right]^{2\alpha+1} + \left[(1-x_k)^{1/2} + \frac{1}{n} \right]^{2\alpha+1} \right\}. \end{aligned} \quad (3.8)$$

We distinguish two cases.

Case 1. $k=0$.

If $(1-x_1)^{1/2} \leq 1/n$ then $\sin(\theta_1/2) \leq 1/n$ and hence $\theta_1 \leq c/n$; for otherwise (3.8) with $k=0$ implies

$$(1-x_1)^{\alpha+1} \leq \frac{c}{n} (1-x_1)^{\alpha+(1/2)},$$

which again gives $(1-x_1)^{1/2} \leq c/n$ and hence $\theta_1 \leq c/n$.

Case 2. $k>0$.

By Claim 1 (3.8) implies

$$(1-x_{k+1})^{\alpha+1} - (1-x_k)^{\alpha+1} \leq \frac{c}{n} [(1-x_{k+1})^{\alpha+(1/2)} + (1-x_k)^{\alpha+(1/2)}],$$

which becomes

$$\sin^{2\alpha+2} \frac{\theta_{k+1}}{2} - \sin^{2\alpha+2} \frac{\theta_k}{2} \leq \frac{c}{n} \left[\sin^{2\alpha+1} \frac{\theta_{k+1}}{2} + \sin^{2\alpha+1} \frac{\theta_k}{2} \right].$$

Thus for $\alpha \geq -1/2$ by Lemma 5 with $d=2\alpha+1$

$$\begin{aligned} \theta_{k+1} - \theta_k & \leq \frac{c(\theta_{k+1} - \theta_k) \left(\sin^{2\alpha+1} \frac{\theta_{k+1}}{2} + \sin^{2\alpha+1} \frac{\theta_k}{2} \right)}{n \left(\sin^{2\alpha+2} \frac{\theta_{k+1}}{2} - \sin^{2\alpha+2} \frac{\theta_k}{2} \right)} \\ & \leq \frac{c(\theta_{k+1} - \theta_k)}{n \left(\sin \frac{\theta_{k+1}}{2} - \sin \frac{\theta_k}{2} \right)} = \frac{2c}{n \cos \theta^*} \leq \frac{c}{n}, \end{aligned}$$

where $\theta^* \in (\theta_k/2, \theta_{k+1}/2)$ and hence $\cos \theta^* \geq c_1$.

If $\alpha < -1/2$ then by Lemmas 3, 4, and 8 with notation (2.13)

$$\begin{aligned} \theta_{kn}(w_1(-(1/2) - \alpha)) &\geq \theta_{kn}(w_1(0)) \geq \theta_{kn}(w_1(-1 - \alpha)) \\ &= \theta_{k-1, n-1}(w_1(m-1 - \alpha)) \\ &\geq \theta_{k-1, n-1}(w_1(-(1/2) - \alpha)) \\ &\geq \theta_{k-1, n}(w_1(-(1/2) - \alpha)) \end{aligned}$$

and hence by the previous conclusion

$$\theta_{k+1, n} - \theta_{kn} \leq \theta_{k+1, n}(w_1(-(1/2) - \alpha)) - \theta_{k-1, n}(w_1(-(1/2) - \alpha)) \leq \frac{c}{n}.$$

Claim 3. $\theta_{k+1} - \theta_k \geq \frac{c}{n}$, $k = 1, 2, \dots, n-1$.

By (2.2) with $d\mu(t) = w(t) dt$, (2.11), and (2.12) for $|x| \leq 1 - (c_1/n^2)$

$$\begin{aligned} |A'_0(\ell_k, x_k; x)| (1-x^2) w^{(\alpha, \beta)}(x) &\leq cn \int_{-1}^1 |A'_0(\ell_k, x_k; t)| (1-t^2)^{1/2} w^{(\alpha, \beta)}(t) dt \\ &\leq cn^2 \int_{-1}^1 A_0(\ell_k, x_k; t) w^{(\alpha, \beta)}(t) dt \\ &\leq cn^2 \int_{-1}^1 A_0(\ell_k, x_k; t) w(t) dt = cn^2 \lambda_{0nm}(w, x_k) \end{aligned}$$

and hence by Claim 2

$$\begin{aligned} |A'_0(\ell_k, x_k; x)| &\leq cn^2 \lambda_{0nm}(w, x_k) (1-x^2)^{-1} w^{(\alpha, \beta)}(x)^{-1} \\ &\leq cn(1-x^2)^{-1/2}, \quad x \in [x_{k+1}, x_k]. \end{aligned}$$

Meanwhile

$$\begin{aligned} 1 &= A_0(\ell_k, x_k; x_k) - A_0(\ell_k, x_k; x_{k+1}) = (x_k - x_{k+1}) A'_0(\ell_k, x_k; \xi) \\ &\leq cn(x_k - x_{k+1})(1 - \xi^2)^{-1/2} \leq cn(x_k - x_{k+1})(1 - x_k^2)^{-1/2}, \end{aligned}$$

that is,

$$x_k - x_{k+1} \geq \frac{c(1-x_k^2)^{1/2}}{n}.$$

Hence

$$(\sin \theta^*)(\theta_{k+1} - \theta_k) \geq \frac{c \sin \theta_k}{n}, \quad \theta^* \in (\theta_k, \theta_{k+1}).$$

Thus by Claim 2 we conclude $\theta_{k+1} - \theta_k \geq \frac{c}{n}$. ■

3.3. Proof of Theorem 3

By Theorems 1 and 2 it suffices to show the sufficiency. (1.17) implies (1.19) and

$$(1 - x_{kn}^2)^{1/2} \sim (1 - x_{kn}^2)^{1/2} + \frac{1}{n}. \quad (3.9)$$

Suppose without loss of generality that $1 > x \geq 0$. Let $x \in [x_{k+1, n}, x_{kn}]$ and $y \in [x_{r+1, n}, x_{rn}]$ with $-1 < y < x$. By (2.9), (1.11), (1.19), and (3.9)

$$\begin{aligned} \int_y^x w(t) dt &\leq \int_{x_{r+1, n}}^{x_{kn}} w(t) dt \leq \sum_{v=k}^{r+1} \lambda_{0vnm}(w) \\ &\leq \frac{c}{n} \sum_{v=k}^{r+1} \left[(1 - x_{vn})^{1/2} + \frac{1}{n} \right]^{2\alpha+1} \\ &\leq c \sum_{v=k}^{r+1} (1 - x_{vn})^\alpha (x_{vn} - x_{v+1, n}) \rightarrow c \int_y^x (1-t)^\alpha dt \end{aligned}$$

as $n \rightarrow \infty$. That is

$$\int_y^x w(t) dt \leq c \int_y^x (1-t)^\alpha dt.$$

Since y is arbitrary, by mean value theorem for integrals we get

$$w(x) \leq c(1-x)^\alpha \leq c(1-x)^\alpha (1-x)^\beta.$$

Similarly, again by (2.9), (1.11), (1.19), and (3.9) as $n \rightarrow \infty$

$$\int_y^x w(t) dt \geq \int_{x_m}^{x_{k+1, n}} w(t) dt \geq \sum_{v=k+1}^r \lambda_{0vnm}(w) \rightarrow c_1 \int_y^x (1-t)^\alpha dt.$$

Hence $w(x) \geq c_1(1-x)^\alpha \geq c_1(1-x)^\alpha (1+x)^\beta$. ■

3.4. Proof of Theorem 4

The first formula in (1.20) directly follows from (1.11) and (2.3); the second one follows from the first one, (1.19), and (2.21). ■

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